ECON B225: ECONOMIC DEVELOPMENT

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The Solow Growth Model

Agenda

- Familiarize you with the mathematical foundation of the Solow Model and why it has the structures it does
- Derive the Solow Model diagram
- Derive the steady state level of capital and output
- Eventually figure out what the Solow Model can tell us about what drives growth

The Solow Growth Model

- **Also called the "neoclassical growth model"**
- **Robert Solow won the Nobel Prize for this in 1988**
  - Prior dominant models were the Harrod-Domar Model, and Lewis Model
  - Responds specifically to Harrod-Domar

- **Goal of the model:**
  - Provide a framework for how growth in GDP per capita occurs over time
  - Provide a theoretical answer to the questions:
    - How does economic growth (development) happen?
    - What are the dynamics causing the growth?

A Quick Note

- **Many students find the Solow Model tough**
- **There are going to be a lot of equations deriving the model in a little bit**
- **It's normal to feel overwhelmed and to need to come back to this multiple times before it all fits together**
  - Come see me in office hours if you need extra help
- **Schaffner presents Solow with time as continuous**
  - This is the traditional presentation
  - I will present it here with <u>discrete time</u>: also the focus of your PS and exam
    - Relate this to Schaffner, but study the framework I'm providing

Starting Point

- **Starting idea: A county's production in a time period $\boldsymbol{t}$, $\boldsymbol{Y_t}$, is the result of some production process $\boldsymbol{F(\cdot)}$**
  - $F(\cdot)$ describes how the country's economy combines factors of production together to produce Y_t
    - What are "factors of production"?
      - Labor, denoted L_t
      - Capital, denoted K_t

- **Production Function: $\boldsymbol{Y_t = F(K_t, L_t)}$**
  - All this is saying is that a county's output in time period t is a function of capital and labor in time period t

Properties of $F(\cdot)$

- **What are desirable properties for $F(\cdot)$?**

- **Property 1:**
  - If there are no workers ($L_t = 0$), what should Y_t be?
  - If there is no capital ($K_t = 0$), what should Y_t be?
  - **So if either argument (or both) is zero, the function should map to 0,**
    - Formally: $F(0, L_t) = 0$, $F(0, K_t) = 0$, and $F(0, 0) = 0$

Properties of $F(\cdot)$

- **What are desirable properties for $F(\cdot)$?**

- **Property 2:**
  - If you hold K_t constant, and increase L_t, what should happen to output?
  - If you hold L_t constant, and increase K_t, what should happen to output?
  - **So $F(\cdot)$ should have positive but diminishing returns to scale in both inputs,**
    - Formally (just for students who like calculus):
      - $\frac{\partial F(K_t,L_t)}{\partial L_t} > 0, and \frac{\partial^2 F(K_t,L_t)}{\partial L_t^2} < 0$
      - $\frac{\partial F(K_t,L_t)}{\partial K_t} > 0, and \frac{\partial^2 F(K_t,L_t)}{\partial K_t^2} < 0$

Properties of $F(\cdot)$

- **What are desirable properties for $F(\cdot)$?**

- **Property 3:**
  - If you increase K_t together with L_t at the same rate, how should Y_t change?
  - **So capital and labor are complements,**
    - formally (again if you like calc),
      - $\frac{\partial^2 F(K_t, L_t)}{\partial K_t \partial L_t} > 0$

Properties of $F(\cdot)$

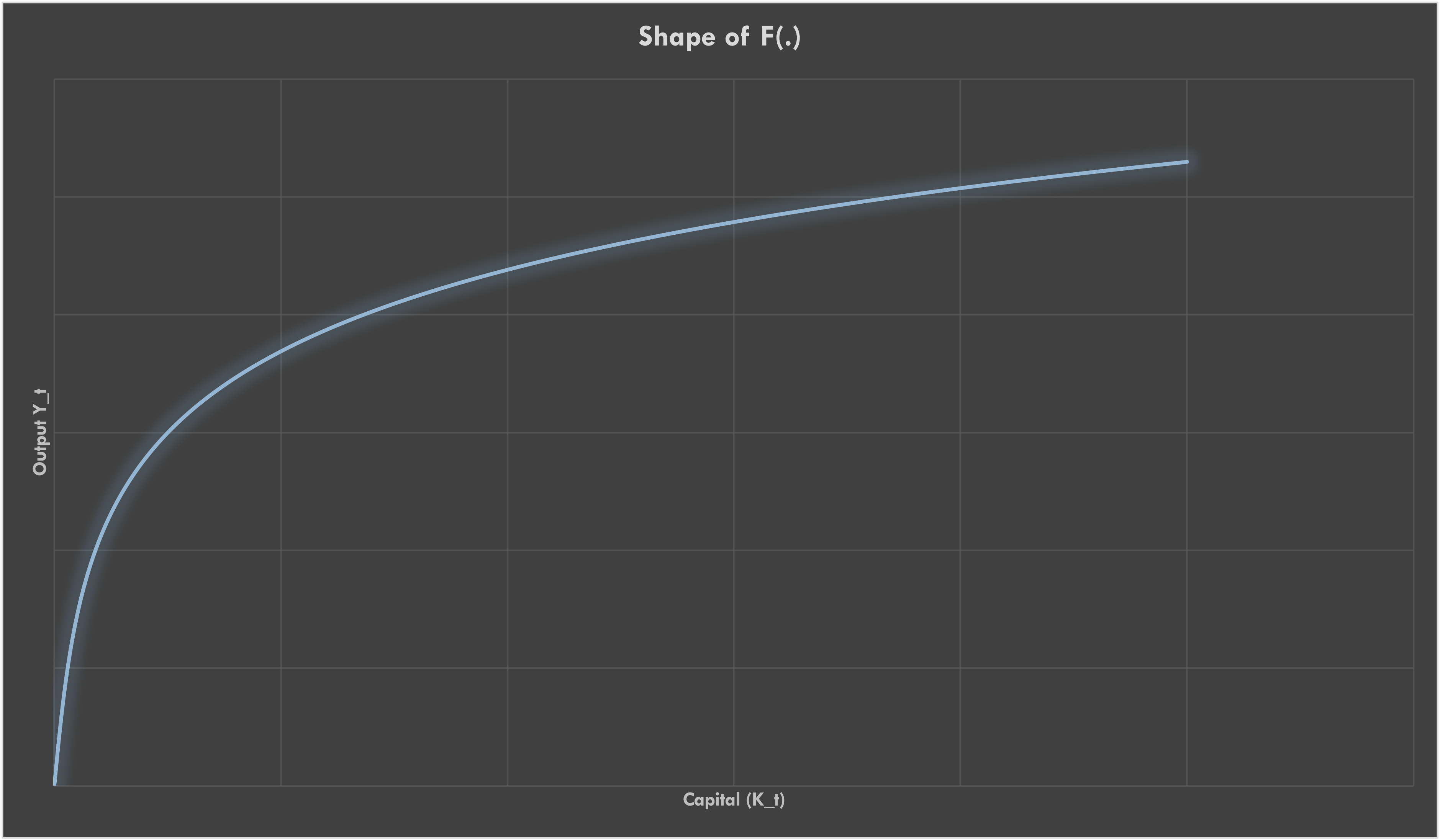
- **What are desirable properties for $F(\cdot)$?**

- **Property 4:**
  - If you double capital and labor, should you be able to more than double your output?
  - No. We if we increase capital and labor by some fixed amount, output should increase by the same amount
  - **The function should show constant returns to scale,**
    - Formally: $\omega Y_t = F(\omega K_t, \omega L_t)$ $where\ \omega\ is\ some\ scalar$

The Shape of $F(\cdot)$

- **Holding L_t constant at $\bar{L}_t$, what should be the shape for $Y_t = F(K_t, \bar{L}_t)$? (in Y_t, K_t space)**
  - Recall your Econ 105 class

The Shape of $F(\cdot)$



The Shape of $F(\cdot)$

- **Holding L_t constant at $\bar{L}_t > 0$, what should be the shape for $Y_t = F(K_t, \bar{L}_t)$? (in Y_t, K_t space)**
  - It should have this large rate of growth at smaller quantities and then smaller rate of growth at larger quantities
  - What production dynamic does this shape represent? What's the intuition? (again draw on 105 knowledge)
    - If workers aren't increasing with the capital then at some point you won't have enough workers to work all the machines and each additional machine will be less productive

So what's the function form of $F(\cdot)$

- **So we know the qualities we want in $F(\cdot)$. What functional form actually achieves this and demonstrates these qualities?**
  - Happily, the Cobb-Douglas Functional Form has all of these qualities
- **Here is the Cobb-Douglas Functional Form,**

$$Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}, where\ \alpha \in (0,1) \tag{1}$$

- **What's still missing here?**
  - What happens to technology over time, what happens to worker's skills over time?
    - Technological change!

Adding Technological Change to the Model

- **We're just going to model technical change as a scaler, A ,to the production function**

$$Y_t = F(K_t, L_t) = A \times K_t^{\alpha} L_t^{1-\alpha}, where\ \alpha \in (0,1) \quad (2)$$

  - If the scaler is larger than 1, how does it affect Y_t?

- **So now we have a simple model of a country's production at any period in time**

What is α?

- **Some may not have seen the Cobb-Douglas functional form in ECON 105**
- **It is just a functional form that has the properties we just described so we can use it in the model to derive its properties easily**
- **You may be wondering how to interpret α**
  - α is the "output elasticity" of capital
  - $(1-\alpha)$ is the "output elasticity" of labor
  - Informally, α is just a measure of how responsive output is to capital, $(1-\alpha)$ is a measure of how responsive output is to labor

What is α?

- **To see this, just take the natural logarithm of both sides of equation (2),**

$$ln(Y_t) = ln(A \times K_t^{\alpha} L_t^{1-\alpha}) \quad (3)$$

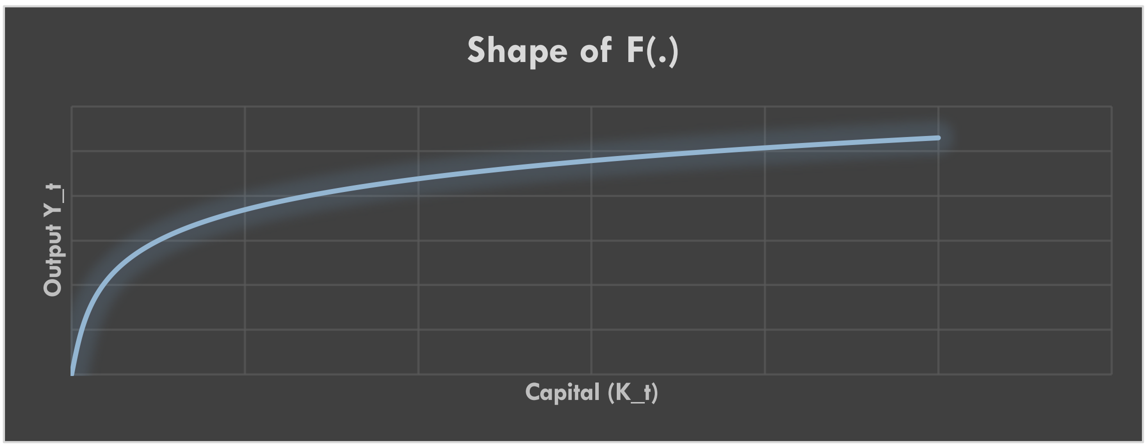
- **By log rules, this is the same as,**

$$ln(Y_t) = ln(A) + \alpha\, ln(K_t) + (1-\alpha)\, ln(L_t) \quad (4)$$

- **So α determines how much log-transformed Y_t changes with log-transformed K_t and L_t**
  - When both variables are logged the slope term represents the percentage change in Y_t as a result of a 100 percent change in K_t or L_t
    - Recall from ECON 105, that elasticities are measured in percentage changes, hence the interpretation of α and $(1-\alpha)$ as "elasticities" of capital and labor!

Review

- Holding L_t constant at $\bar{L}_t > 0$, what does the shape for $Y_t = F(K_t, \bar{L}_t)$ (in Y_t, K_t space) represent? What's the story?



- What does α represent in the Solow Model Production Function?
- What's the Solow Model trying to describe?

Accounting for Saving Between Time Periods

- **This is a "dynamic" model, so it has a time dimension**
- **This means we need to think carefully about how capital changes over time**
  - Suppose a country receives some capital stock by creating a machine. What happens to that machine over time? Will it stay as pristine and perfectly operational as it is on day 1? (No! It breaks down and becomes less productive)
  - This deterioration in the productive capacity of capital is called "depreciation"

Accounting for Saving Between Time Periods

- **This is a "dynamic" model, so it has a time dimension**
- **This means we need to think carefully about how capital changes over time**
  - But what can a country do to make sure it always has enough capital in each time period for production?
  - It can add to the capital stock in the next time period from what it produces in this time period, some portion of Y_t
  - This is called "investment"

Intuition for the Equation of Motion of Capital

- **With these concepts of "depreciation" and "investment" we can come up with a simple statement about how capital changes over time in a country**
- **If a country in time period t has K_t as its capital stock, what should be the capital stock in time period $t+1$, denoted K_{t+1}?**
  - Answer: The original capital stock minus the amount of depreciation plus what the country invested!

Deriving the Equation of Motion of Capital

- **So putting that into math results in the following,**

$$K_{t+1} = K_t - \delta K_t + I_t, where\ \delta \in (0,1) \tag{6}$$

  - Here, δ is the depreciation rate, and I_t is the **amount of production** in the time period that was **saved** and invested into the capital stock in the next period

- **So we can be even more detailed and we can say,**

$$I_t = sY_t\ where\ s \in (0,1) \tag{7}$$

  - Here, s is the savings rate

Other Important Assumptions

- **We'll also assume that the country can either consume or invest output, so**

$$Y_t = I_t + C_t \tag{8}$$

  - Here, C_t stands for consumption
  - We won't use this much today, but it will be useful in your problem set

- **Population growth rate is $\boldsymbol{n \in (0, 1)}$**

Deriving the Equation of Motion of Capital

- **Let's go back to equation (6) and work with it,**

$$K_{t+1} = K_t - \delta K_t + I_t, where \ \in (0,1) \qquad (9)$$

- **Let's substitute in our definition of $\boldsymbol{I_t = sY_t}$**

$$\Longrightarrow K_{t+1} = K_t - \delta K_t + sY_t \qquad (10)$$

- **Now, let's move K_t over to the LHS,**

$$\Longrightarrow K_{t+1} - K_t = sY_t - \delta K_t \qquad (11)$$

- **This is the "Equation of Motion of Capital"**

$$\Delta K = sY_t - \delta K_t \qquad (12)$$

Summary: We Have Two Useful Structures

- **The Production Function:**

$$Y_t = F(K_t, L_t) = A \times K_t^{\alpha} L_t^{1-\alpha}, where\ \alpha \in (0,1) \quad (13)$$

- **The Equation of Motion of Capital:**

$$\Delta K = sY_t - \delta K_t \quad (14)$$

How These Equations Work Together

- **Suppose:** $\boldsymbol{\alpha = \frac{1}{2}, s = .666, \delta = .333, A = 1, n = .333}$
- $Y_t = F(K_t, L_t) = A \times K_t^{\alpha} L_t^{1-\alpha}$
- $K_{t+1} = K_t - \delta K_t + sY_t$

$\boldsymbol{t}$	$\boldsymbol{K_t}$	$\boldsymbol{L_t}$	$\boldsymbol{Y_t}$	$\boldsymbol{K_{t+1}}$
1	9	9	$= 9^{\frac{1}{2}} 9^{\frac{1}{2}} = 9$	$= 9 - (.333)(9) + (.666)(9) = \boxed{12}$
2	12	$= (9)(1.333) = 12$	$= 12^{\frac{1}{2}} 12^{\frac{1}{2}} = 12$	$= 12 - (.3)(12) + (.666)(12) = 16$
3				
4				

Summary: We Have Two Useful Structures

- **The Production Function:**

$$Y_t = F(K_t, L_t) = A \times K_t^{\alpha} L_t^{1-\alpha}, where\ \alpha \in (0,1) \quad (13)$$

- **The Equation of Motion of Capital:**

$$\Delta K = sY_t - \delta K_t \quad (14)$$

- **Remember the goal: A model for *per capita* growth over time**
  - How do we adapt what we have to measure ***per capita*** production and ***per capita*** change in capital over time?

Assumption that Population = L_t

- **We'll assume that a country's population is equal to its labor force (L_t)**
  - Wait! That's obviously not true! Why?
  - Many people in the population do not or can't work

- **While this assumption is not true, it is "benign" in that it is easy to adapt the model to account for this fact**
  - When you do, it just introduces a scalar that doesn't affect any of the dynamics of the model or change any of its conclusions
  - So we (and every textbook) just go with this assumption because it makes the math a little easier

Deriving the per capita production function

- **Okay, So lets use the relationship that Population $= L_t$. What's per capita production?**

$$\frac{Y_t}{L_t} = \frac{A \times K_t^{\alpha} L_t^{1-\alpha}}{L_t} \tag{15}$$

$$\Longrightarrow \frac{Y_t}{L_t} = A \times K_t^{\alpha} L_t^{1-\alpha} L_t^{-1} = A \times K_t^{\alpha} L_t^{1-\alpha-1} \tag{16}$$

$$\Longrightarrow \frac{Y_t}{L_t} = A \times K_t^{\alpha} L_t^{-\alpha} = A \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{17}$$

Deriving the per capita production function

- **Okay, So lets use the relationship that Population = L_t. What's per capita production?**

$$\Longrightarrow \frac{Y_t}{L_t} = A \times K_t^{\alpha} L_t^{-\alpha} = A\left(\frac{K_t}{L_t}\right)^{\alpha} \qquad (18)$$

- **$\frac{K_t}{L_t}$ is just capital per capita, we'll define, $\frac{K_t}{L_t} = k_t$, so**

$$\Longrightarrow \frac{Y_t}{L_t} = \boxed{y_t = Ak_t^{\alpha}} \qquad (19)$$

What's the Equation of Motion per capita?

- **We'll do the same trick. Here's the equation of motion again,**

$$K_{t+1} - K_t = sY_t - \delta K_t \tag{20}$$

- **Let's put each term in per capita terms**

$$\Delta k = \frac{K_{t+1}}{L_{t+1}} - \frac{K_t}{L_t} = \frac{sY_t}{L_t} - \frac{\delta K_t}{L_t} \tag{21}$$

$$\Longrightarrow \Delta k = sy_t - \delta k_t \tag{22}$$

  - You'll often see Δk notated with $\dot{k}$ or $\frac{\partial k}{\partial t}$ in a continuous framework (as opposed to a discrete framework that I'm presenting)

Building the Solow Model Diagram

- **We're first going to use this equation to derive a statement for the "rate of per capita capital accumulation"**

$$\frac{\Delta k}{k_t} = \underbrace{\frac{K_{t+1} - K_t}{K_t}}_{\text{Rate of capital accumulation}} - \underbrace{\frac{L_{t+1} - L_t}{L_t}}_{\text{Rate of Labor Force Growth (Population Growth Rate)}} \qquad (23)$$

This is just $n \in (0, 1)$

Building the Solow Model Diagram

- Rearranging,

$$\frac{\Delta k}{k_t} + n = \frac{K_{t+1} - K_t}{K_t} \tag{24}$$

- Using equation (20) The RHS of equation (24) is just equal to $\frac{sY_t - \delta K_t}{K_t}$, by substitution then

$$\frac{\Delta k}{k_t} + n = \frac{sY_t}{K_t} - \delta \tag{25}$$

Building the Solow Model Diagram

- **Subtracting n from both sides and multiplying through by k_t,**

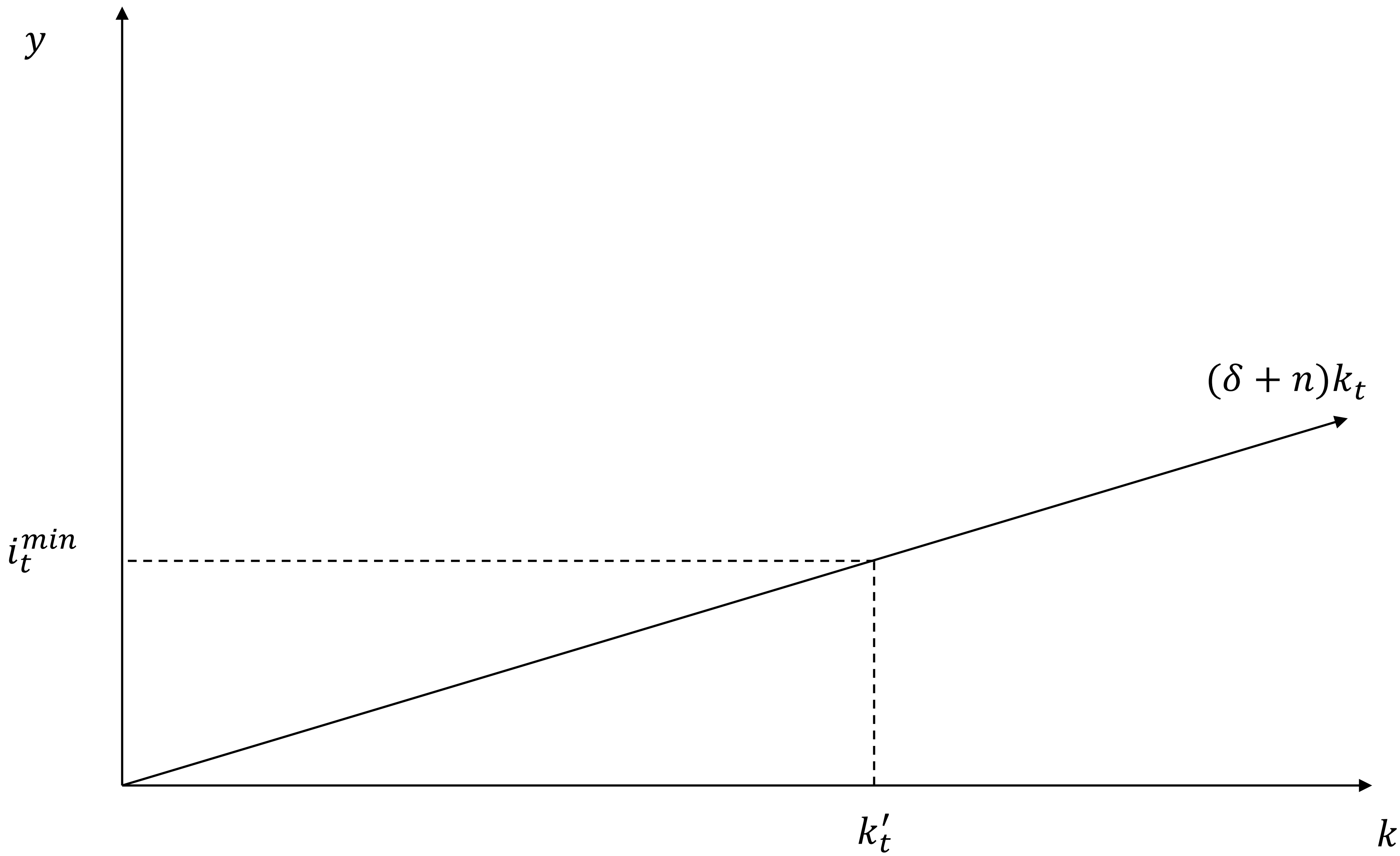
$$\Delta k = sy_t - (n + \delta)k_t \tag{26}$$

- **If we set $\Delta k = 0$, so the rate of capital accumulation is 0,**

$$sy_t = \boxed{(n + \delta)k_t} \tag{27}$$

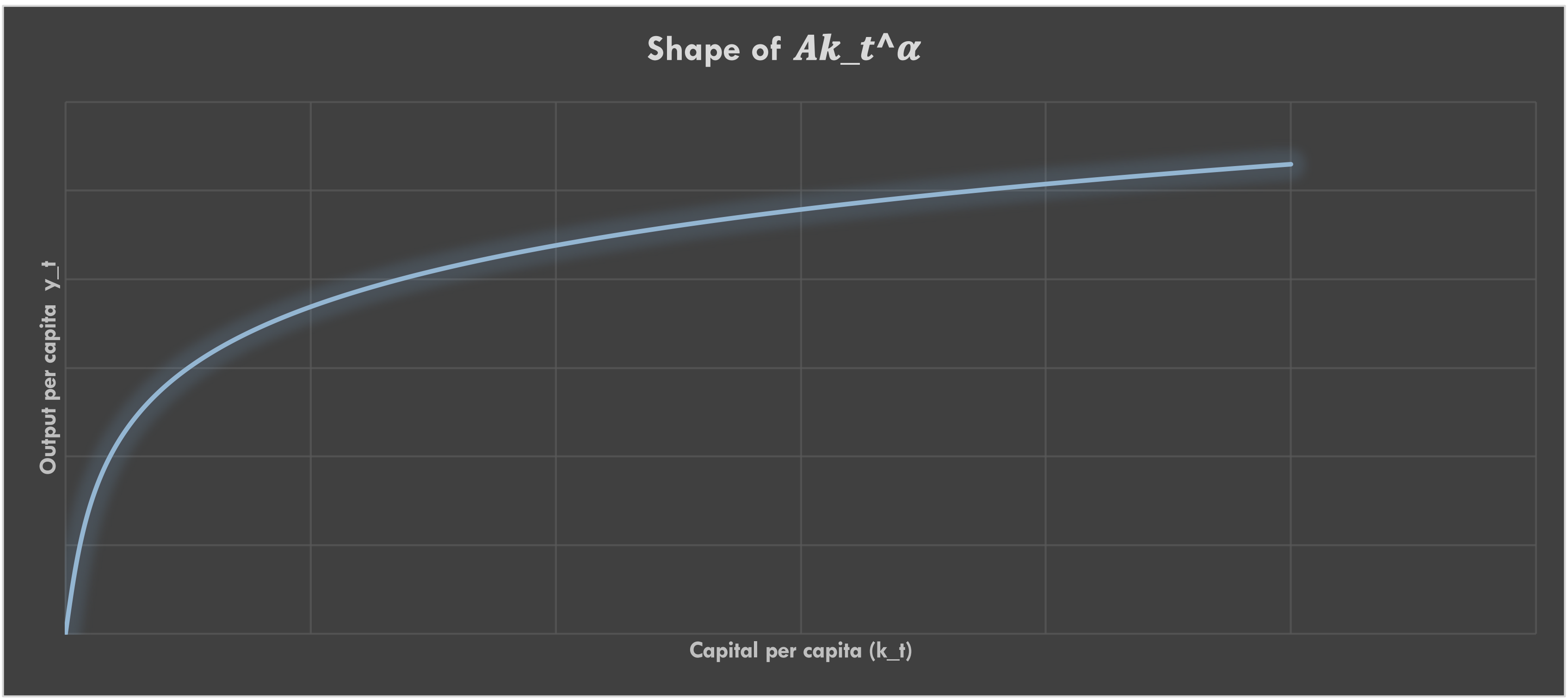
Minimum investment to keep capital stock per capita constant over time i_t^{min}

Building the Solow Model Diagram

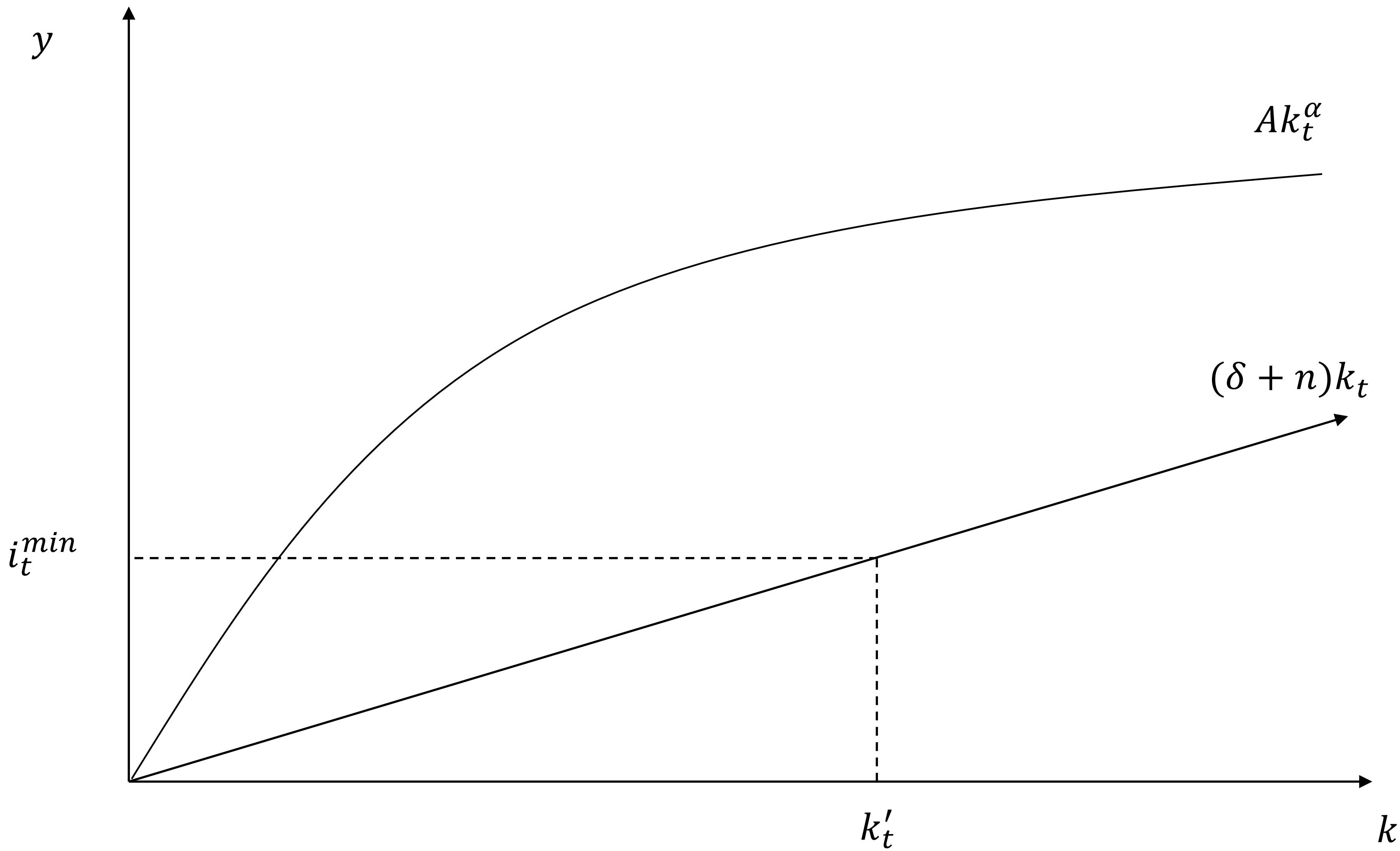


Adding Production to the Diagram

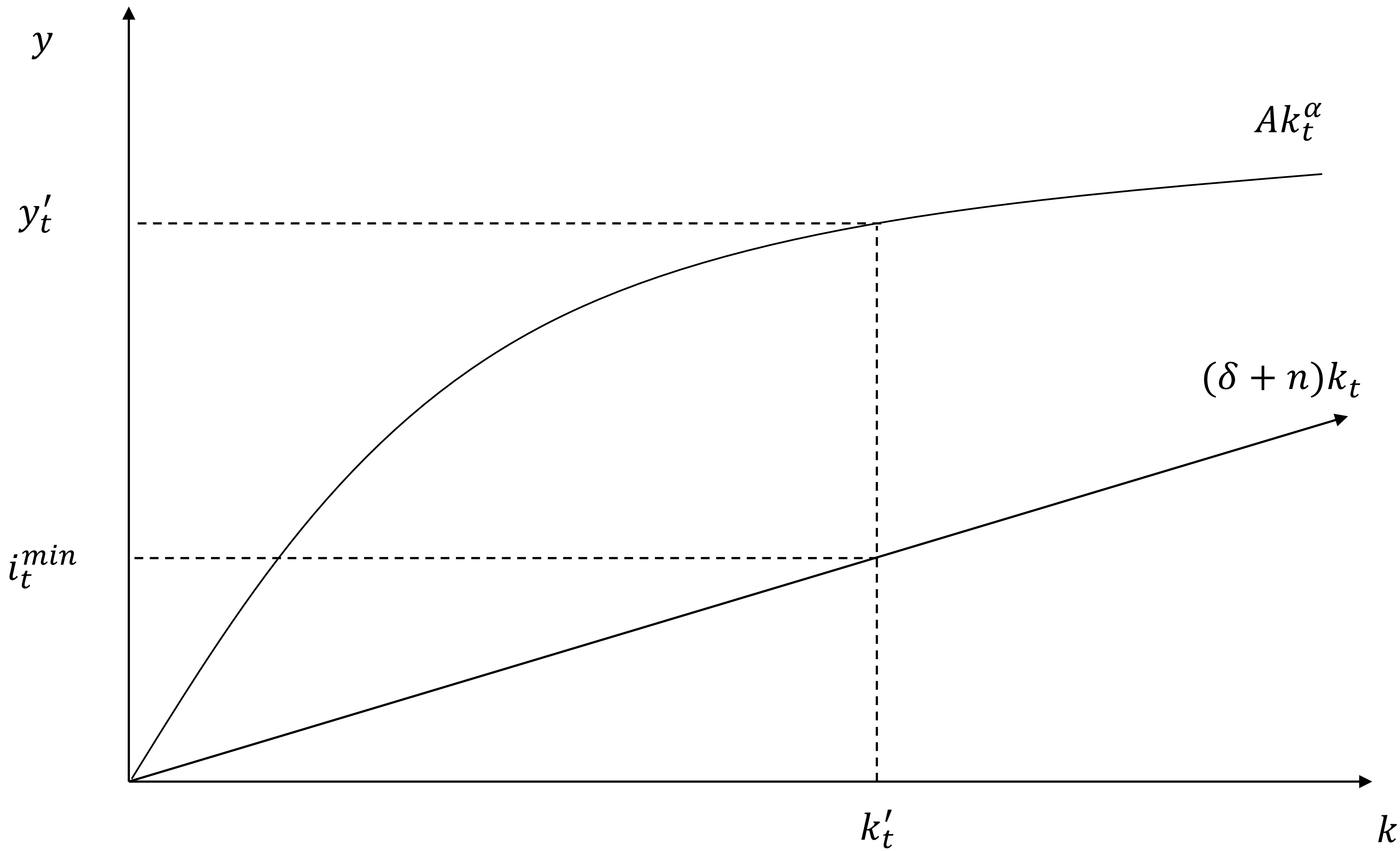
- Now let's use $\frac{Y_t}{L_t} = y_t = Ak_t^{\alpha}$, and plot this in the same space
- What shape will it have? Remember that $\alpha \in (0, 1)$



Building the Solow Model Diagram



Building the Solow Model Diagram



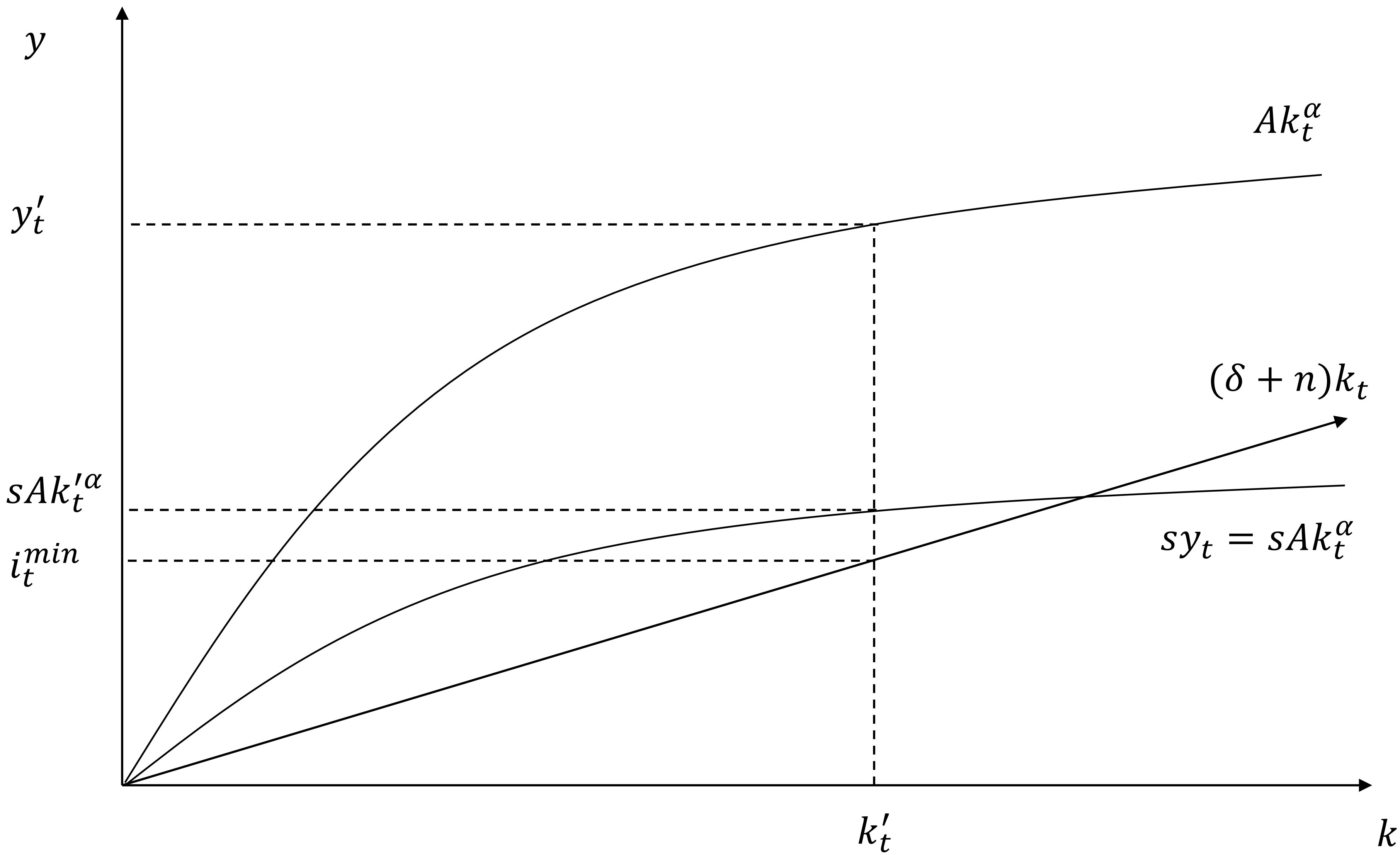
Adding Savings to the Diagram

- So how do we add savings to the diagram? Well, we can use the fact that $y_t = Ak_t^{\alpha}$

$$sy_t = sAk_t^{\alpha} \tag{28}$$

- Remember that s is just a scalar between 0 and 1
- We just plotted Ak_t^{α}. So how would sAk_t^{α} show up in the diagram?

The Solow Model Diagram

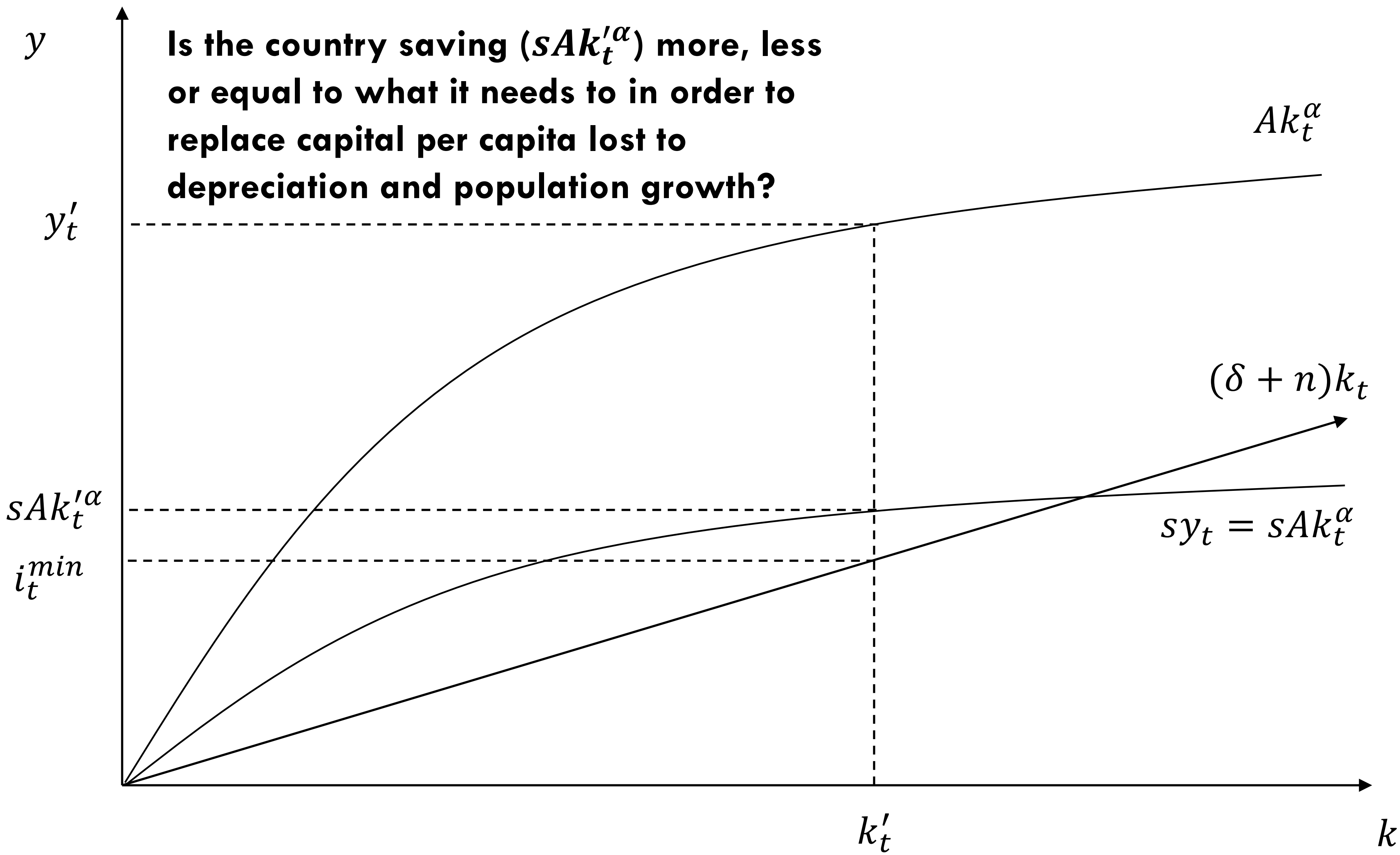


Deriving The Behaviors of the Model

- So that is the Solow Model Diagram
- Now, we’re going to examine it to figure out how it suggests growth will occur in a country
- There are two (2) major implications to cover

Consider k_t'. What is Happening to the Country's Capital Stock?

Is the country saving ($sAk_t'^{\alpha}$) more, less or equal to what it needs to in order to replace capital per capita lost to depreciation and population growth?

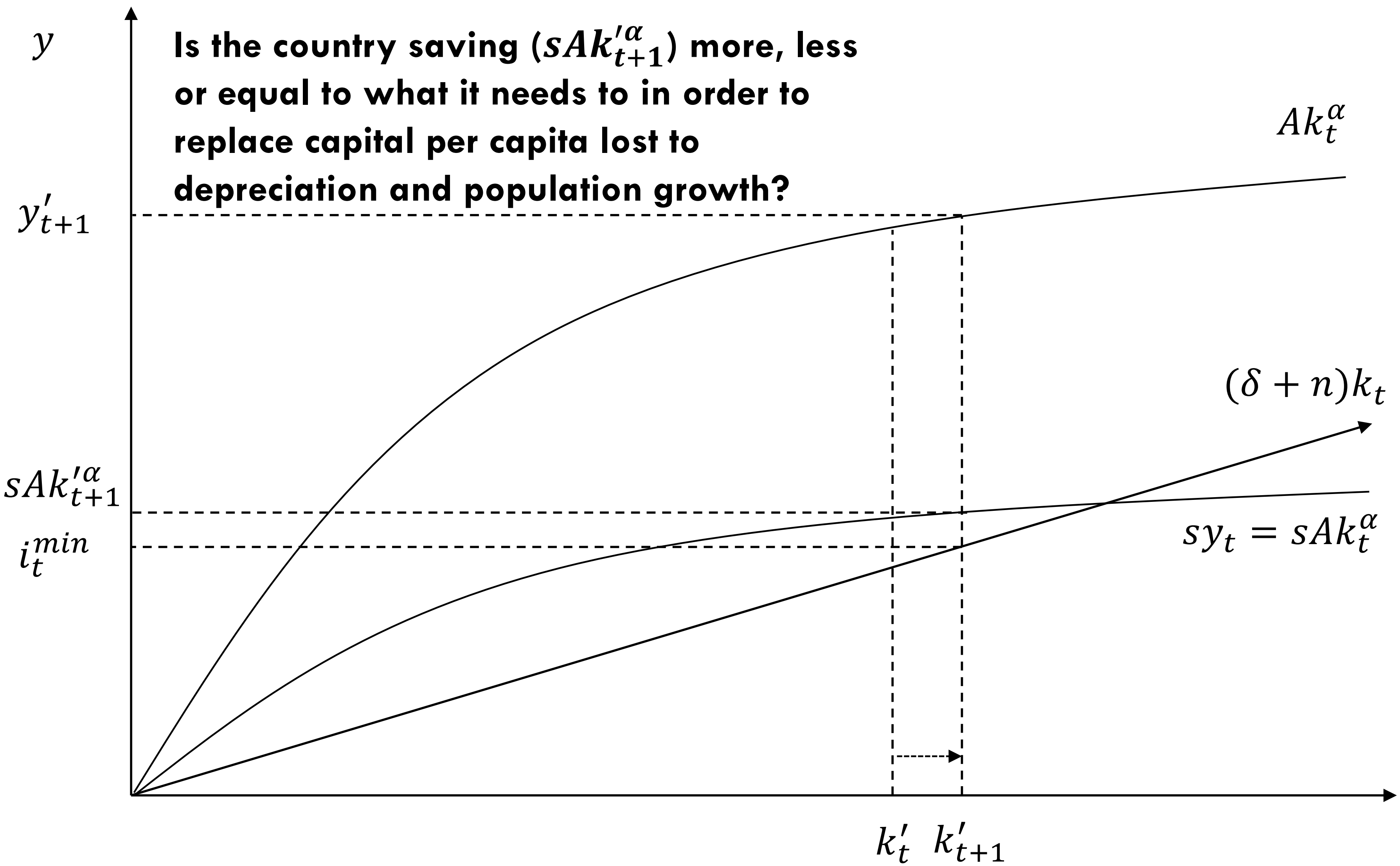


Consider k_t'. What is Happening to the Country's Capital Stock?

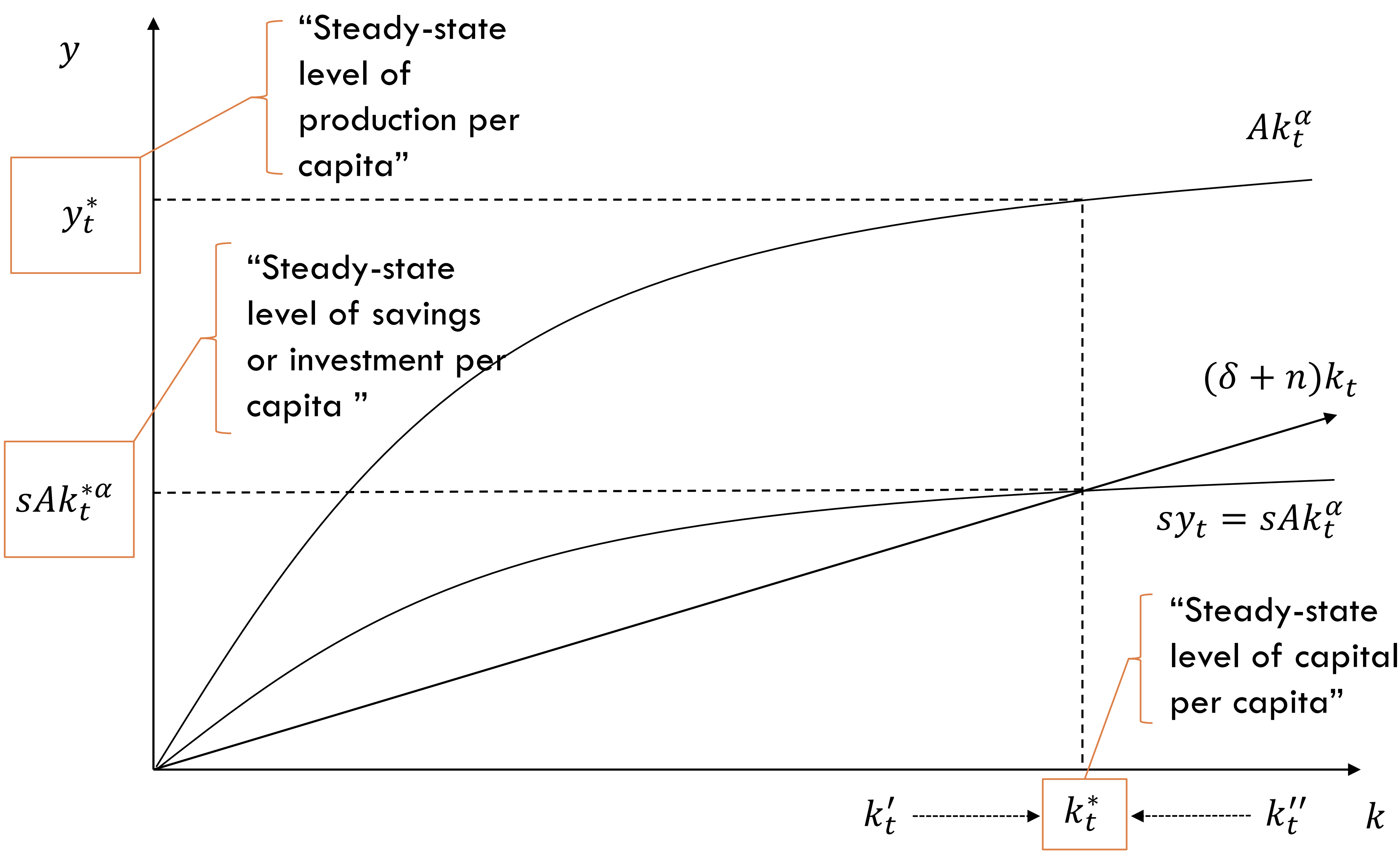
- **At k_t', $sAk_t'^{\alpha} > \delta k_t$, it is saving more than what it needs to just replace capital per capita lost to depreciation**
  - So is the capital stock per capita growing, shrinking or staying the same over time at k_t'?
    - It's growing!
    - So where will k_{t+1}' be on the chart relative to k_t'?

Consider k'_{t+1}. What is Happening to the Country's Capital Stock?

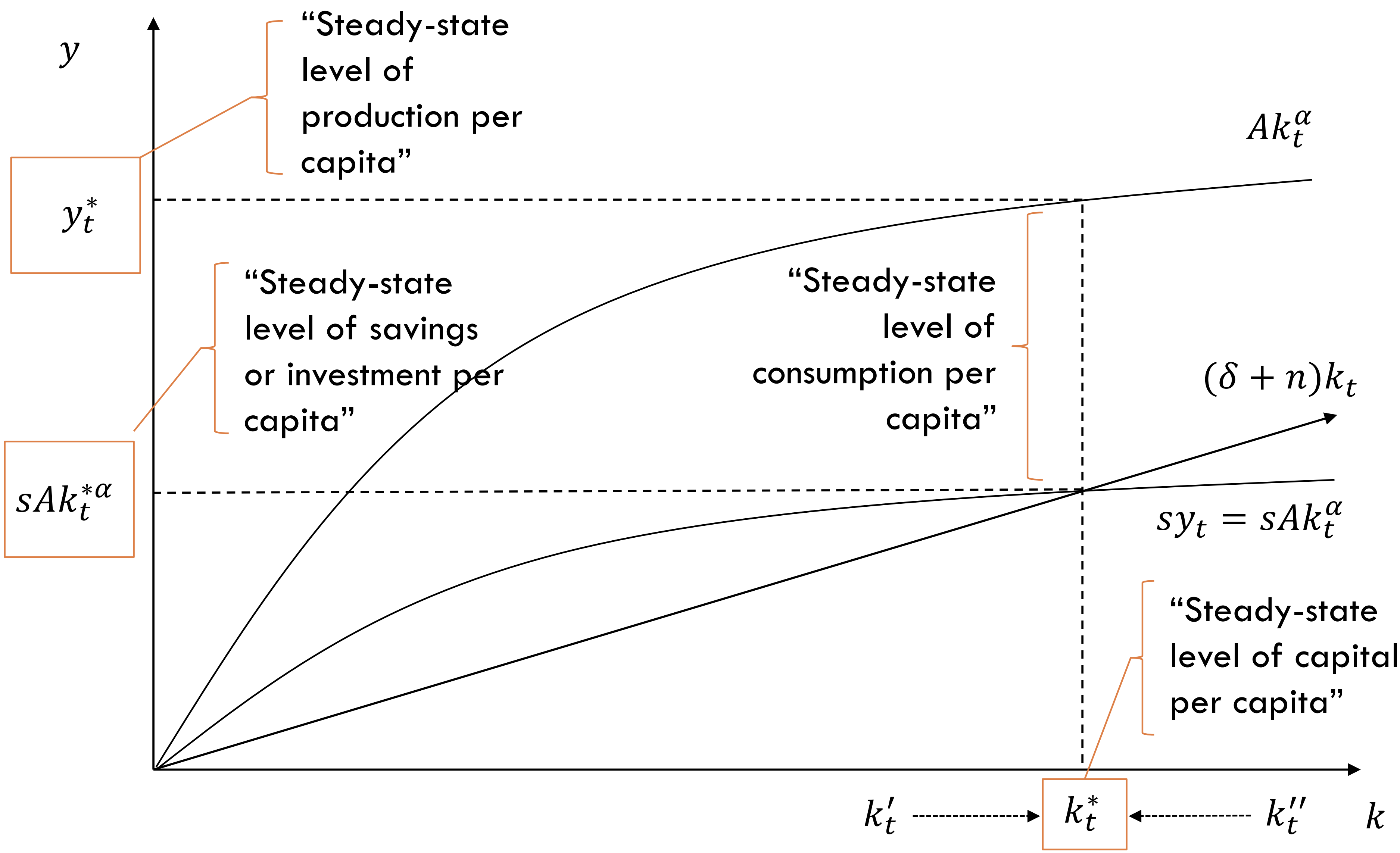
Is the country saving (sAk'^{α}_{t+1}) more, less or equal to what it needs to in order to replace capital per capita lost to depreciation and population growth?



This Process Stops at $(n + \delta)k_t = sy_t$



What is Consumption per Capita in the Steady State?



Solow Model Implication 1

- **So, at some point, the economy can no longer grow through capital accumulation, this is defined where,**

$$(\delta + n)k_t = sy_t \tag{29}$$

Model Simulation Exercise

- Okay, let's simulate this model and see if it conforms this
- Suppose the following:
  - $A = 1$
  - $s = 0.2$
  - $\delta = 0.05$
  - $\alpha = \frac{1}{2}$
  - $k_{t=1} = 10$
- What is $\boldsymbol{y_{t=1}}$? What is $\boldsymbol{k_{t=2}}$? Remember,

$$k_{t+1} = k_t - \delta k_t + s y_t \tag{30}$$

$$y_t = A k_t^{\alpha} \tag{31}$$

Model Simulation Exercise

- First, you calculate the output in $t = 1$. This is just plug and chug with equation (31):

$$y_{t=1} = Ak_{t=1}^{\alpha} = (1)\left(10^{1/2}\right) = 3.16 \tag{32}$$

- Then, you use that result, and the givens to calculate the level of capital in $t = 2$. Again, this is just a plug-and-chug exercise:

$$k_{t=2} = k_{t=1} - \delta k_{t=1} + sy_{t=1} \tag{33}$$

$$\Longrightarrow k_{t=2} = 10 - (0.05)(10) + (0.2)(3.16) \tag{34}$$

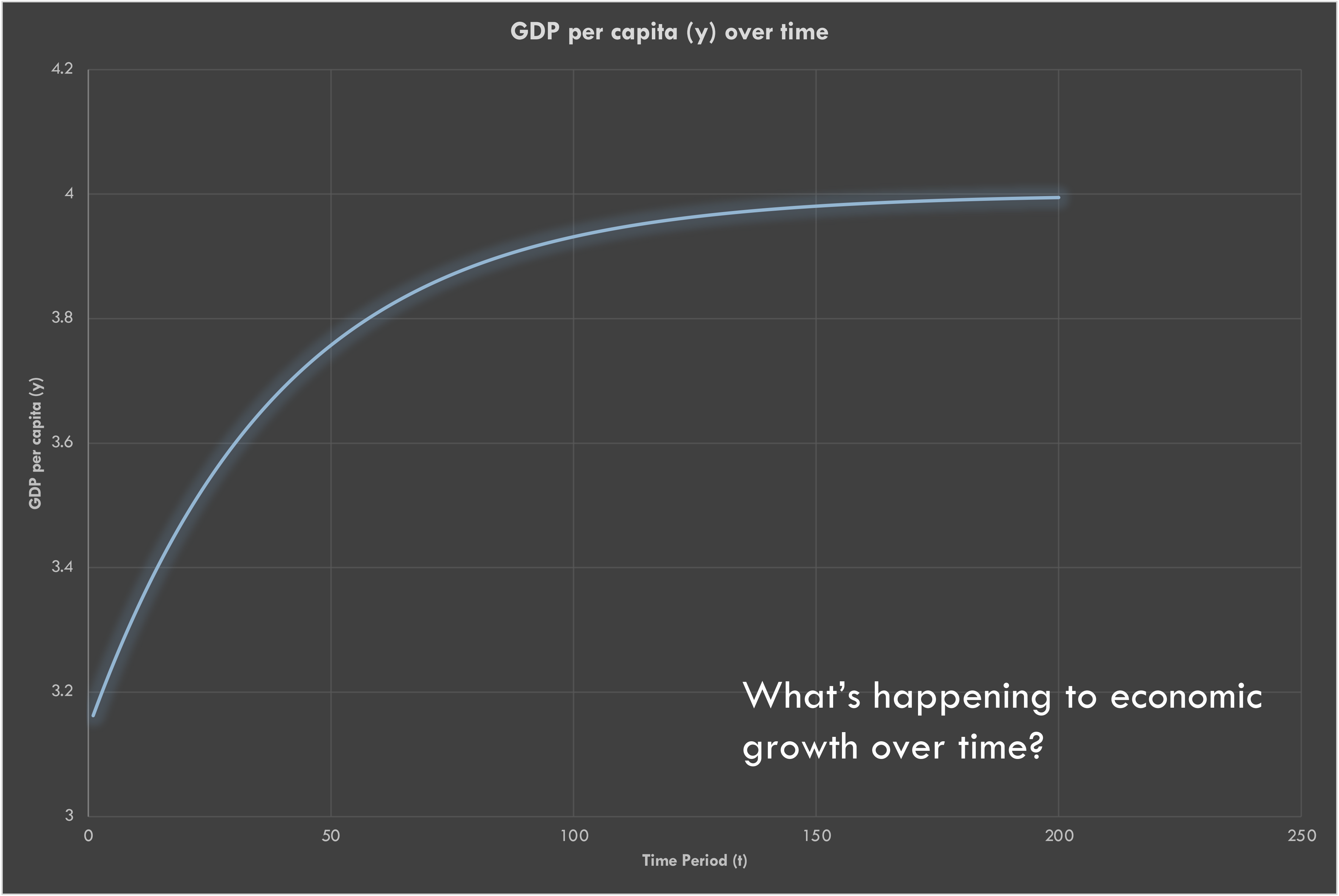
$$\Longrightarrow k_{t=2} = 10.132 \tag{35}$$

The Next Time Periods

- What is $y_{t=2}$?
- What is $k_{t=3}$?

- $y_{t=2} = 3.183$
- $k_{t=3} = 3.204$

- Let's look at what happens when we calculate out the values for the first 200 time periods and graph it

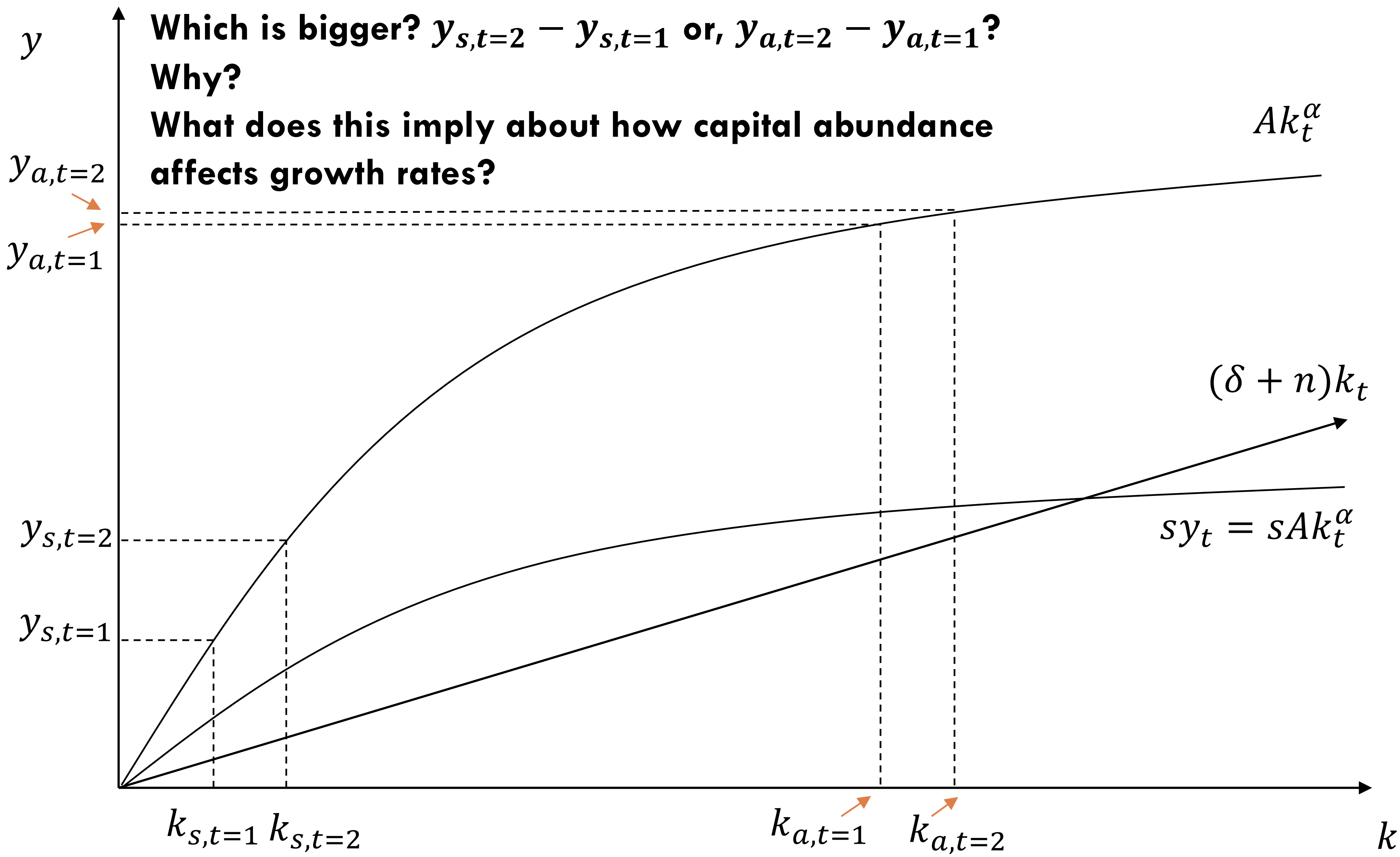

GDP per capita (y) over time
GDP per capita (y)
4.2
4
3.8
3.6
3.4
3.2
3
0
50
100
150
200
250
Time Period (t)
What's happening to economic growth over time?

Growth Comparisons between Capitol-scarce and Capital-abundant Countries

Which is bigger? $y_{s,t=2} - y_{s,t=1}$ **or,** $y_{a,t=2} - y_{a,t=1}$**?**

Why?

What does this imply about how capital abundance affects growth rates?

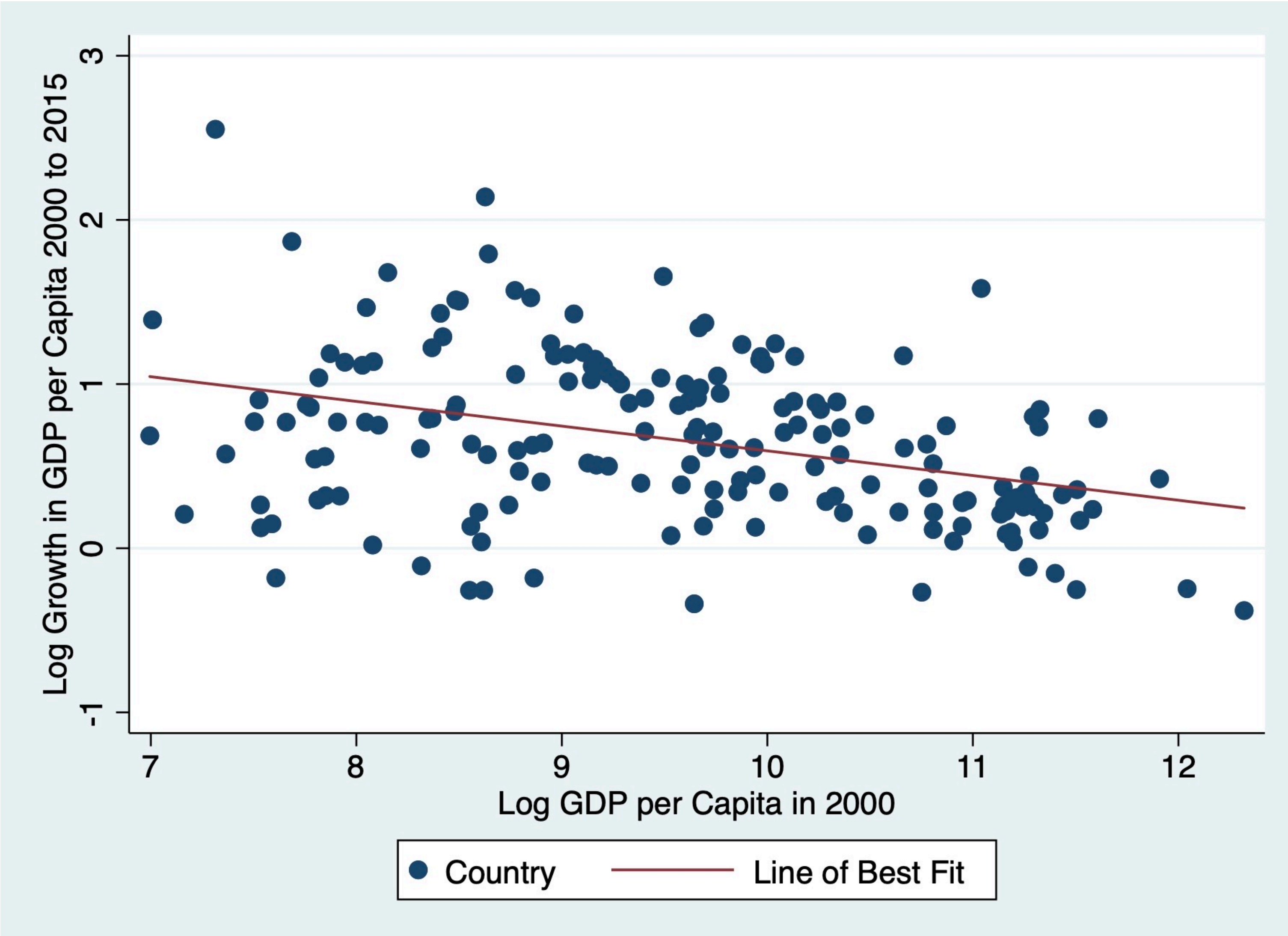


Solow Model Implication 2

- Capital-scarce countries will grow faster than capital abundant countries, all else constant
- In other words, poorer countries will grow faster than richer ones, all else constant
- Formally (just for students who like calculus),

$$k_{a,t} > k_{s,t} \Longrightarrow \frac{\partial y_{a,t}}{\partial t} < \frac{\partial y_{s,t}}{\partial t} \tag{36}$$

We Actually See this in Empirical Analysis



- Instructor's Calculations
- Data Source: Penn World Tables 10.0
- This is a publicly available dataset. Feel free to download it and work with it yourself (in Stata or Excel): https://www.rug.nl/ggdc/productivity/pwt/?lang=en

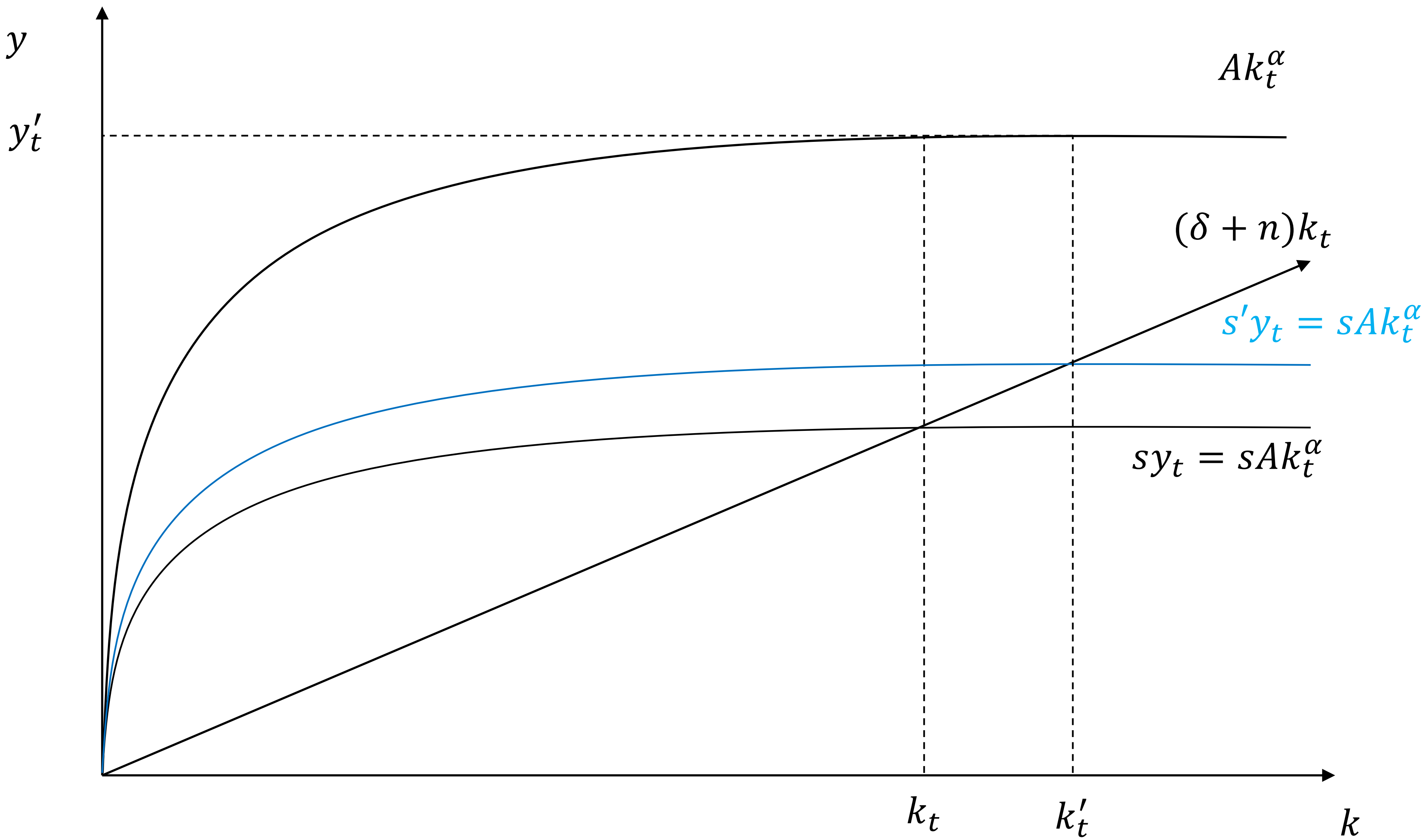
Some Group Work

- **So… as a country grows through accumulating capital, its growth rate slows, and eventually goes to zero**
  - This means growth should completely stop in the future!
- **Get into groups**
  - Let's say you are leading a country that wants economic growth, but you've hit your steady state equilibrium
  - <u>How could you intervene in the process described in the Solow Model to get growth going again?</u>
    - Hint: Look at each variable or parameter and ask how it might be influenced

Ideas?

- **Manipulate the savings rate?**
  - Let's look at how this would affect the chart

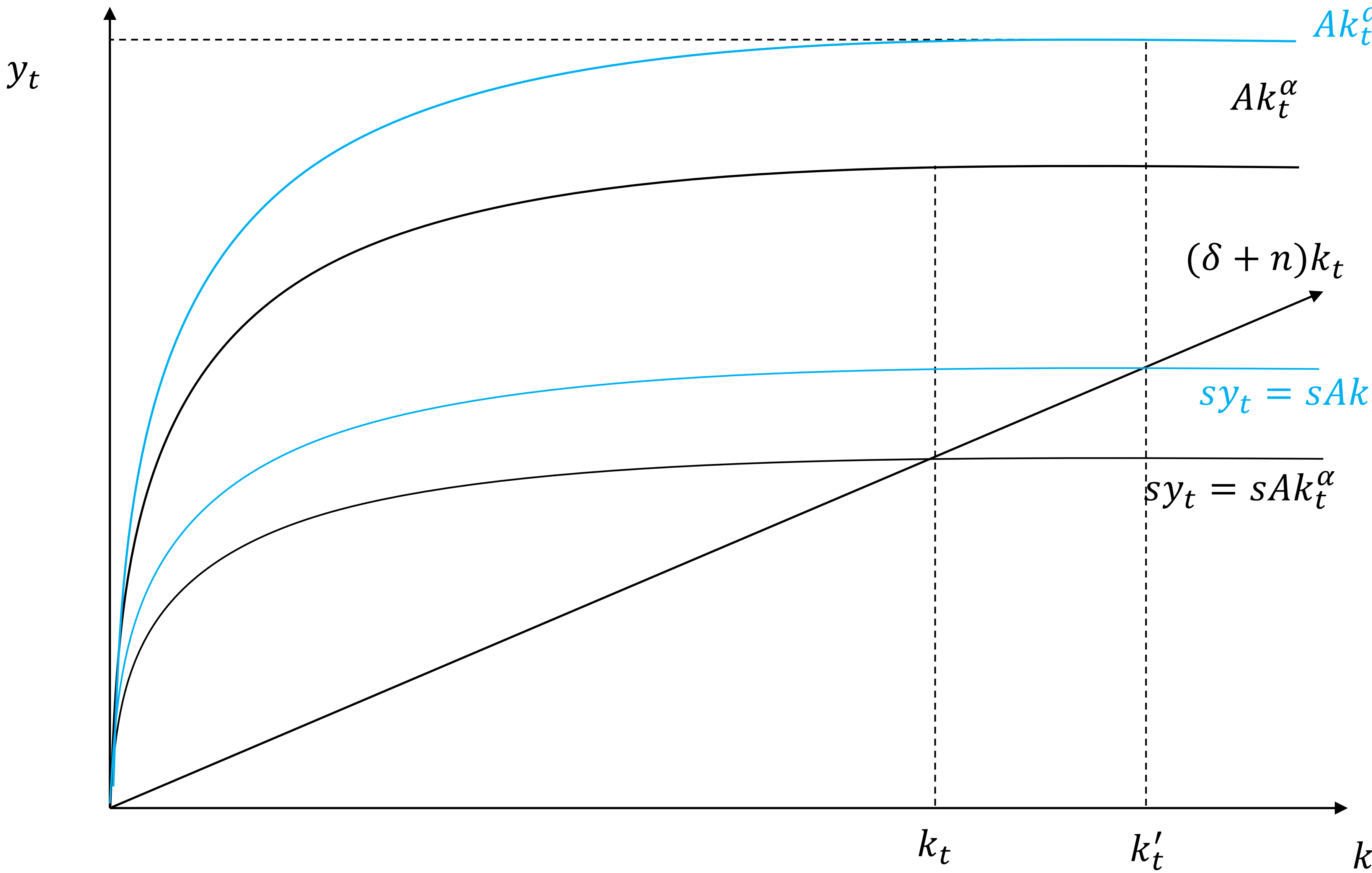
Adjusting the Savings Rate If Already on the Flat Part of the Curve



Ideas?

- **Manipulate the savings rate?**
  - Let's look at how this would affect the chart
  - This just pushes the steady state down the road
  - Eventually you'll run out of options, because s has an upper bound
  - So this is only a temporary option
- **Manipulating α to be bigger?**

Adjusting the Savings Rate If Already on the Flat Part of the Curve



Ideas?

- **Manipulate the savings rate?**
  - Let's look at how this would affect the chart
  - This just pushes the steady state down the road
  - Eventually you'll run out of options, because s has an upper bound
  - So this is only a temporary option
- **Manipulating α to be bigger?**
  - This makes output bigger with capital
  - It'll raise both the Ak_t^α and the sAk_t^α curves but will maintain their basic shapes
  - α also has an upper bound, so this again just pushes the steady state problem down the road

More Ideas?

- **Manipulate δ somehow**
  - Hard to see how you'd do this
  - Also a temporary solution that would just delay the arrival of the steady state
- **This leaves A**
  - A represents technological enhancement that makes capital and labor more productive
  - Called **"Total Factor Productivity" or TFP**
- **How might a policy maker or leader decide to influence TFP?**

"Endogenous" Growth Models: TFP can be determined by countries

- **If this is influenceable by society and its leaders/institutions, how can we amend Solow to reflect this**
  - Right now A is a simple scaler. Maybe we could turn it into a dynamic variable (changes over time)
    - Perhaps a share of investment goes to capital accumulation and a share goes to developing new technology through R and D
  - This is the idea behind "endogenous" growth models
    - Always results in the violation of one of the four desirable qualities of the production function we discussed

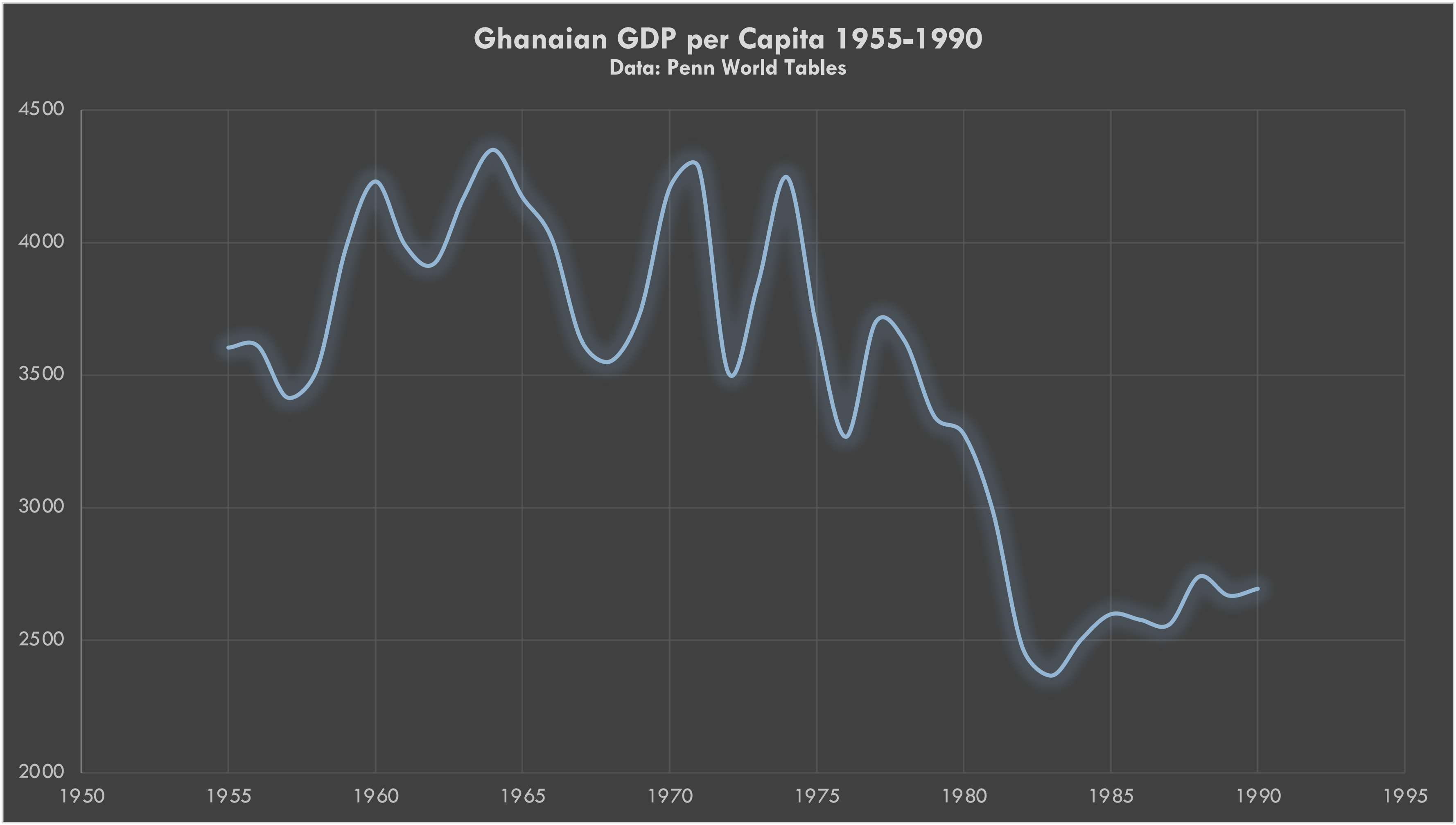
So TFP is the Key According to Solow

- **So what is TFP?**
  - Very hard to measure
  - It's conceptually anything that enhances the production process of a country to be more efficient (to produce more per worker)
- **What might achieve A TFP increase?**
  - Ideas:
    - Literal technological innovation and scientific breakthroughs
    - Institutional reform
    - More wide use of already available technologies
    - Increased educational levels (more efficiency in taking on tasks and more capacity for scientific innovation)
    - More healthy population
    - Improved infrastructure and production-facilitating public goods

Conclusions

- **So Solow explains that sustained growth can only happen through technological progress**
  - Capital accumulation alone only drives growth up to a point
- **Suggests that poor countries will grow faster than rich countries**
- **However, it leads to the obvious question: If A (TFP) is really the key, what are its components and dynamics?**
  - Many, many hypotheses for that

Challenges: Growth in Reality for Many Countries Not So Smooth and Positive



Need for a Short-Term Model

- **Ghana's growth looks nothing like what the Solow Model would predict**
  - Motivates a need for a new model to explain short-term periods where growth might stall, or the economy might contract
- **The Australian Model provides this**